

# Boundary-value problems in cosmological dynamics

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The dynamics of cosmological gravitating system is governed by the Euler and the Poisson equations. Tiny fluctuations near the big bang singularity are amplified by gravitational instability into the observed structure today. Given the current distribution of galaxies and assuming initial homogeneity, dynamical reconstruction methods have been developed to derive the cosmic density and velocity fields back in time. The reconstruction method described here is based a least action principle formulation of the dynamics of collisionless particle (representing galaxies). Two observational data sets will be considered. The first is the distribution of galaxies which is assumed to be an honest tracer of the mass density field of the dark matter. The second set is measurements of the peculiar velocities (deviations from pure Hubble flow) of galaxies. Given the first data set, the reconstruction method recovers the associated velocity field which can then be compared with the second data set. This comparison constrains the nature of the dark matter and the relation between mass and light in the Universe.

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## I. INTRODUCTION

Cosmology is concerned with observing and modeling the universe on large scales: from our own Milky Way, other galaxies, galaxy clusters, super clusters up to the largest scales as probed by measurements of the cosmic microwave background radiation (CMB). These observations span a huge range of scales and all strongly suggest that: 1) the dominant form of matter is dark (a factor of 6 in mass over the normal baryonic matter), 2) the clustering amplitude decreases with scale, and 3) structure forms by gravitational amplification of tiny initial fluctuations. These are some of the main component of the standard paradigm in cosmology. Violation of any of them or all of them is consistent with only a very limited set of observations, if any. Cosmology has had a great impact on other fields of physics and science in general. The sheer existence of the gravitationally dominant dark matter has stimulated scientists' (and others') vivid imagination for a few decades now. Abundance and masses of non-standard particles have been constrained from the observed clustering pattern alone. In addition to gravity, hydrodynamical processes can greatly influence the formation and evolution of galaxies, groups and clusters of galaxies. Hydrodynamical effects, however, play a minor role in shaping the observed distribution of galaxies on scales a few times larger than the size of galaxy clusters. Therefore, gravitational instability theory directly relates the present-day large scale structure to the initial density field and provides the framework within which the observations are analyzed and interpreted. Gravitational instability is a non-linear process. Analytic solutions exist only for configurations with special symmetry, and approximate tools are limited to moderate density contrasts. So, numerical methods are nec-

essary for a full understanding of the observed large scale structure of the universe. There are two complementary numerical approaches. The first approach relies on  $N$ -body techniques designed to solve an initial value problem in which the evolution of a self-gravitating system of massive particles is determined by numerical integration of the Newtonian differential equations. Combined with semi-analytic models of galaxy formation,  $N$ -body simulations have become an essential tool for comparing the predictions of cosmological models with the observed properties of galaxies. Because the exact initial conditions are unknown, comparisons between simulations and observations are mainly concerned with general statistical properties. The second approach aims at finding the past orbits of mass tracers (galaxies) from their observed present-day distribution. The orbits must be such that the initial spatial distribution is homogeneous. This approach is very useful for direct comparisons between different types of observations of the large scale structure. Most common are the velocity-velocity (hereafter  $v$ - $v$ ) comparisons between the observed peculiar velocities of galaxies and the velocity field inferred from the galaxy distribution in redshift surveys. This type of analysis yield the cosmological mass density parameter  $\Omega_m$ . Any systematic mismatch between the fields serves as an indication to the nature of galaxy formation and/or the origin of galaxy intrinsic scaling relations used to measure the distances, provided that errors in the calibration have been properly corrected for. This second approach also allows to perform back-in-time reconstructions of the density field on scales  $\sim 5 \text{ h}^{-1} \text{ Mpc}$  [3].

Finding the orbits that satisfy initial homogeneity and match the present-day distribution of mass tracers is a boundary value problem. This problem naturally lends itself to an application of Hamilton's variational principle where the orbits of the objects are found by searching for stationary variations of the action subject to the boundary conditions. The use of the Principle of Least Action in a cosmological framework has been pioneered by

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Peebles (1989) [6] and has long been restricted to small systems such as the Local Group [7] and the Local Supercluster [8]. Early applications to large galaxy redshift surveys have been hampered by the computational cost of handling the relatively large number of objects. Subsequent numerical applications speeded up the method and allowed the reconstruction of the orbits of  $\sim 10^3$  particles (Shaya, Peebles & Tully 1995). However, it was only recently that the improvement of the minimization techniques and the use of efficient gravity solvers made it possible to deal with more than  $10^4$  objects[5], comparable to the number of objects contained in the largest all-sky galaxy catalogs.

## II. COSMOLOGICAL DYNAMICS

For the background cosmology we work with a Friedmann-Robertson-Walker Universe. In this uniform background, the physical distance,  $r$ , between two points is  $r \propto a(t)$  where  $a(t)$  is the scale factor. We consider a matter dominated universe with mean density  $\bar{\rho} = \Omega\rho_c$  with  $\rho_c = 3H^2/8\pi G$ . For a  $\Omega = 1$ , we get a critical density flat universe with  $a \sim t^{2/3}$ . The Universe is geometrically open for  $\Omega < 1$  and close for  $\Omega > 1$ . Current observations indicate that the Universe contain a cosmological constant which makes it flat even though  $\Omega \approx 0.3$ [9]. Apart from the dependence of  $a$  on  $t$  the presence of a cosmological constant has very little effect on our description here. In particular, the equations of motion of perturbations remain correct. We further define,  $H(t) = \dot{a}/a$  is the Hubble function, and denote the comoving coordinate of a patch of matter by  $\mathbf{x} = \mathbf{r}/a$ . The fluctuations are described by the density contrast  $\delta(\mathbf{x}, t) = \rho(\mathbf{x}, t)/\bar{\rho}(t) - 1$  and the comoving velocity by  $\mathbf{v} = d\mathbf{x}/dt$ . Also, let  $D(t)$  be the linear density growing mode normalized to unity at the present epoch, and  $f(\Omega_m) = d\ln D/d\ln a \approx \Omega_m^{0.6}$  (e.g., Peebles 1980). The equations governing the evolution of fluctuations in a collisionless mass component in an expanding Universe are, The Euler equation,

$$\frac{d\mathbf{v}}{dt} + 2H\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \varphi_g, \quad (1)$$

the continuity,

$$\frac{\partial \delta}{\partial t} + \nabla \cdot (1 + \delta)\mathbf{v} = 0 \quad (2)$$

and the Poisson equation,

$$\nabla^2 \varphi_g = 4\pi G \bar{\rho} \delta. \quad (3)$$

The term  $2H\mathbf{v}$  in the Euler equation is due to the expansion of the cosmological background. The source term in the Poisson equation represents density fluctuations above the mean background density.

### A. Linear gravitational instability

Neglecting the non-linear terms  $\mathbf{v} \cdot \nabla \mathbf{v}$  and  $\nabla \cdot \delta \mathbf{v}$ , the equations of motion reduce to

$$\delta = -\frac{1}{f(\Omega)H} \nabla \cdot \mathbf{v} \quad (4)$$

and

$$\ddot{\delta} + 2H\dot{\delta} = \frac{3}{2}H^2\Omega\delta, \quad (5)$$

where an over-dot indicates a time derivative. For a critical density Universe ( $\Omega = 1$  and  $H = 2/3t$ ), the equation (5) gives  $\delta_1 \propto t^{2/3}$  and  $\delta_2 \propto t^{-1}$ , as the growing and decaying solutions, respectively. A few things to note. First, without the term  $2H\dot{\delta}$  the solutions would be exponential functions rather than power laws in time. Second, even in the linear regime, the decaying mode prevents a full recovery of the initial conditions, at  $t \approx 0$  near the big bang cosmological singularity. Indeed, recovering this mode requires a precise knowledge of the present  $\delta$  and  $\dot{\delta}$  (or  $\mathbf{v}$ ), in order to prevent a blow-up as  $t \rightarrow 0$ .

The relation (4) has a simple interpretation. Since  $H \sim 1/t$  and  $\nabla^2 \phi_g \sim -\delta$ , it gives the intuitive relation  $\mathbf{v} \sim -\nabla \phi_g t$  between the acceleration,  $-\nabla \phi_g$  and velocity. The relation has played a prominent role in the analysis of large scale structure. The density contrast  $\delta(\mathbf{x})$  as estimated from the distribution of galaxies, could be used in this relation to obtain the associated peculiar velocity  $\mathbf{v}(\mathbf{x})$ . This velocity fields could be compared with the actual observed velocities of galaxies. A good agreement between the fields yields the cosmological density parameter,  $\Omega$ , and also a confirmation of the gravitational instability mechanism for structure formation. But, perhaps more interestingly, any mismatch between the fields could be an indication of strange mode of galaxy/structure formation the result of which is a galaxy distribution different from that of the dark matter.

### B. Non-linear cosmological dynamics

Linear theory is valid only when the fluctuations are small. In practice this is achieved by smoothing the observed galaxy distribution on small scales ( $\lesssim 10\text{Mpc}$ ). We describe here some non-linear methods which can be used for a variety of purposes, e.g. recovery of the initial conditions, estimating  $\mathbf{v}$  from the galaxy distribution and constraining the masses of galactic halos. Here we focus on the estimation of  $\mathbf{v}$ . One can use numerical simulations of non-linear gravity to calibrate semi-analytical non-linear generalizations to (4). The approach is useful as it provides partial differential equations which can be solved for  $\mathbf{v}$  for a given source term,  $\delta$ . Nevertheless, such generalizations are usually statistical in nature. In the following, we will describe a more rigorous and accurate approach.

We switch to a Lagrangian description for a system of  $N$  equal mass particles in an expanding universe. Each particle represents a patch of matter which, for practical purposes, could be a galaxy. The equations of motion are ( $i = 1 \cdots N$ ),

$$\frac{d\mathbf{v}_i}{dt} + 2H\mathbf{v}_i = \mathbf{g}_i, \quad (6)$$

where  $\mathbf{g} = -\nabla\phi_g$  and is given by

$$\mathbf{g}(\mathbf{x}) = -\frac{G}{a^3} \sum_i \frac{\mathbf{x} - \mathbf{x}_i}{|\mathbf{x} - \mathbf{x}_i|^3} + \frac{4}{3}G\bar{\rho}a\mathbf{x}, \quad (7)$$

The equations can be derived from the action,

$$S = \int_0^{t_0} dt \sum_i \left\{ \frac{a^2}{2} \mathbf{v}_i^2 + \frac{G}{a} \left[ \sum_{j < i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} + \frac{2\pi}{3} \bar{\rho} a^3 \mathbf{x}_i^2 \right] \right\} \quad (8)$$

under stationary first variations of the orbits that leave  $\mathbf{x}$  fixed at the present epoch and satisfy the constraint  $t^{1/3}\mathbf{v} \rightarrow \text{const}$  as  $t \rightarrow 0$  [1, 5, 6]. The second condition on the velocities guarantees homogeneity near the big bang singularity  $t \rightarrow 0$ , preventing a blow up of the solutions. We expand the orbits in a time dependent base functions  $q_n(t)$  in the form,

$$\mathbf{x}_i(t) = \mathbf{x}_{i,0} + \sum_{n=1}^{n_{max}} q_n(t) \mathbf{C}_{i,n}, \quad (9)$$

where  $\mathbf{x}_{i,0}$  is the position of the particle  $i$  at the present epoch, and the vectors  $\mathbf{C}_{i,n}$  are the expansion coefficients with respect to which the action is varied, i.e., they satisfy  $\partial S / \partial \mathbf{C}_{i,n} = 0$ . The base functions  $q_n$  are chosen such that the boundary conditions are satisfied.

Our strategy is to find orbits that are as close as possible to the Hubble flow. Therefore, we search for the minimum of the action and do not look for stationary points which might describe oscillatory behavior of the orbits. To find the coefficients  $\mathbf{C}_{i,n}$  that minimize the action, we use the Conjugate Gradient Method (CGM) which is fast and easy to implement. The gravitational force  $\mathbf{g}$  and its potential are computed using the TREECODE gravity solver. The time integration in the expression for the action is done using the Gaussian quadrature method with 10 points at the time abscissa. The CGM requires an initial guess for  $\mathbf{C}_{i,n}$ . We will use the term FAM, for Fast Action Method, to refer to the reconstruction method described here. In the standard FAM application we compute the initial guess using the linear theory relation between the velocity and mass distribution. The minimum of the action proved to be rather insensitive to the choice of initial guess for  $\mathbf{C}_{i,n}$ , as we have checked by running FAM experiments with initial  $\mathbf{C}_{i,n}$  both set to zero and to random numbers with appropriate variance.

Besides the initial set of  $\mathbf{C}_{i,n}$ , the other free parameters are the softening used by the gravity solver and the tolerance parameter that sets the convergence of the CGM method. The success of the least action reconstruction method is illustrated in figure (1).

### III. DISCUSSION

The rapid rotation of galactic disks revealed the existence of dark matter halos which engulf the luminous component. The measured virial motions of galaxies in clusters of galaxies also require the a gravitationally dominant dark component. Away from bound systems of galaxies and galaxy clusters, field galaxies show coherent flow pattern which deviates from a pure Hubble expansion. This coherent velocity field is a direct probe of the large scale dark matter distribution in as much as rotational speeds and virial motions are a measure of the dark matter in galaxies and clusters. Indeed, the cosmic gravitational field responsible for the motions of galaxies, mainly depends on the gravitationally dominant mass density field of the dark matter. The actual distribution of galaxies may well be quite different from the dark matter distribution. Recent analysis of the galaxy surveys, however, reveal a good match between the statistical properties of the galaxy distribution and the corresponding properties for the dark matter as inferred from numerical simulations of dark matter evolution in the universe. This is encouraging, but there may still be significant deviations between the distribution of the dark and luminous components, which are not reflected in statistical comparisons. The only way to detect such deviations is via direct detailed comparisons between the measured velocities of galaxies and velocities estimated from the galaxy distribution. These comparisons have been done in the linear regime. The overall agreement between the fields is impressive, but minor persisting mismatch is detected in some regions in the local volume. It is possible that non-linear analysis at the least action principle could mitigate some of the disagreement. This remains to be seen. The least action principle could also be used to recover the initial conditions [2], allowing us to answer one of the fundamental question of whether or not initial fluctuations were gaussian [4].

The program is not without flaws. Many physical effects need to be addressed in detail. Most pressing is incorporating the assembly (or merging) history of galaxies. Galaxies reside in dark matter halos which form in a hierarchical manner from small to large. Thus our own Milky Way galaxy for example, is likely to have had a major merging activity some 8 Gyr ago. All reconstruction methods assume that galaxies are point tracers of the mass density field and do not account for merging effects.

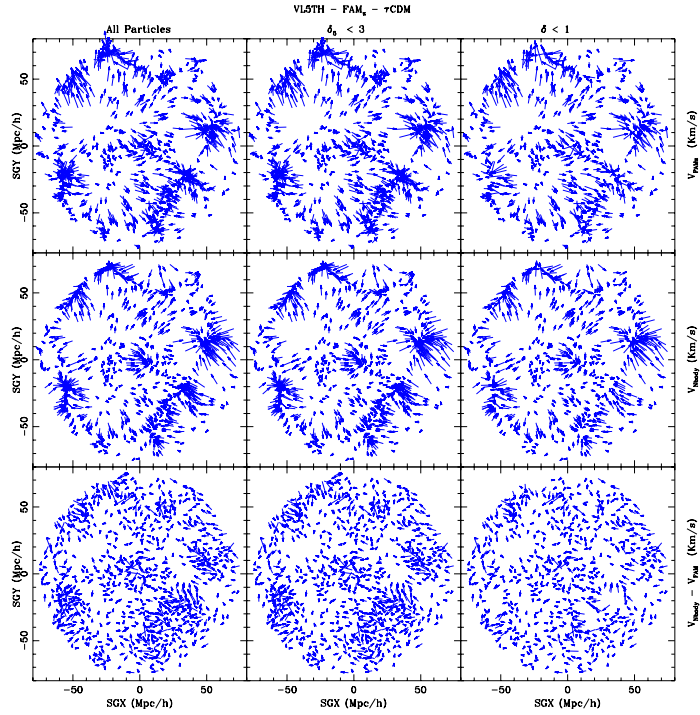


FIG. 1: Maps of 2D-projected peculiar velocities for points residing in a slice of thickness  $6 \text{ h}^{-1} \text{Mpc}$  cut through a simulated catalog. The length of the vectors is drawn in units of  $1 \text{ h}^{-1} \text{Mpc} = 50 \text{ Km s}^{-1}$ . The top row shows the least action predicted velocities (labeled FAMz).  $N$ -body velocities are shown in the middle row. The velocity residuals,  $\mathbf{v}_{Nbody} - \mathbf{v}_{FAMz}$ , are displayed on the bottom. The maps shown in the panels to the left hand side refer to all the points in the slice while only the velocities of points with moderate density contrast are plotted in the central and right columns.

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